AEGC2019
Data to Discovery
Australasian Exploration Geoscience Conference
2-5 September 2019 • Perth, Western Australia
Incorporating the AIG, ASEG, PESA and WABS

Response of a permeable ellipsoid to an imposed magnetic gradient – implications for borehole measurements of the magnetic gradient tensor inside magnetic bodies

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SUMMARY

This paper discusses the internal field and gradient tensor components for a permeable ellipsoidal body in a magnetic medium with an applied non-uniform magnetic field. The formulae can also be applied to the field and gradients measured in an ellipsoidal cavity within a magnetic medium. A long, narrow cylindrical cavity, representing a borehole that penetrates a magnetic body, is a particularly useful special case. I give explicit expressions for the internal gradient tensor within a permeable triaxial ellipsoid placed in a uniform gradient. Expressions are given for the shielding of applied gradients of all orders for the special case of a spherical body and for the amplification of applied gradients of all orders in a spherical cavity within a magnetic medium.

Magnetic sensors cannot measure the internal field and gradient components of a magnetic body directly, as the measuring instrument must be placed within a cavity, such as a borehole, within the magnetic material. This modifies the measured field and gradients. I give expressions for correcting measured gradient tensor components in a borehole within a magnetic body.

Key words: borehole magnetics, magnetic gradient tensor, ellipsoid.

INTRODUCTION

For homogeneous ellipsoidal bodies it is well known that a uniform applied field induces a uniform magnetisation, which produces a uniform demagnetising field (Stratton, 1941; Emerson et al., 1985; Clark et al., 1986). Similarly, it can be shown that within a homogeneous permeable ellipsoid an applied gradient tensor produces a non-uniform demagnetising field associated with a gradient tensor of the same rank. The resultant internal field is a modified version of the imposed non-uniform field that arises from nearby external sources and regional trends.

Recent developments in borehole measurements of the magnetic gradient tensor (e.g. Hillan et al., 2012; Leslie et al., 2015; Sui et al., 2017; Liu et al., 2019) promise to improve 3D interpretations of the subsurface magnetisation distribution. However, correct interpretation of the total magnetic environment, including neighbouring sources, from data acquired in boreholes that penetrate into magnetic bodies requires correct modelling of the internal field of intersected bodies. Furthermore, since measurements are actually made in a cavity (such as a borehole) within the magnetic body, the cavity effect must be taken into account.

MAGNETIC SCALAR POTENTIAL OF AN ARBITRARY NON-UNIFORM APPLIED FIELD

This paper considers magnetostatic fields in linear, piecewise-homogeneous, non-conductive media. For this case, magnetic fields are derivable from a scalar potential Ω. The magnetic scalar potential is only defined to within an arbitrary additive constant, so for simplicity Ω can be set to zero at the origin of co-ordinates. Then in terms of Cartesian co-ordinates, in the vicinity of the origin the applied magnetic scalar potential Ω_0 associated with distant sources at a point $\mathbf{r} = (x, y, z)$ is given by a Taylor's series expansion:

$$\Omega_0(x,y,z) = \left(\frac{\partial\Omega}{\partial x}\right)x + \left(\frac{\partial\Omega}{\partial y}\right)y + \left(\frac{\partial\Omega}{\partial z}\right)z + \frac{1}{2}\left[\left(\frac{\partial^2\Omega}{\partial x^2}\right)x^2 + 2\left(\frac{\partial^2\Omega}{\partial x\partial y}\right)xy + 2\left(\frac{\partial^2\Omega}{\partial x\partial z}\right)xz + \left(\frac{\partial^2\Omega}{\partial y^2}\right)y^2 + 2\left(\frac{\partial^2\Omega}{\partial y\partial z}\right)yz + \left(\frac{\partial^2\Omega}{\partial z^2}\right)z^2\right] + \cdots + \frac{1}{n!}\left[x\left(\frac{\partial}{\partial x}\right) + y\left(\frac{\partial}{\partial y}\right) + \left(\frac{\partial}{\partial z}\right)\right]^n \Omega + \cdots, \qquad (1)$$

where all the partial derivatives are evaluated at the origin.

Using the multinomial theorem, this can be written:

$$\Omega_0(x,y,z) = -(H_0)_x x - (H_0)_y y - (H_0)_z z - \frac{1}{2}\left[G_{xx}x^2 + 2G_{xy}xy + 2G_{xz}xz + G_{yy}y^2 + 2G_{yz}yz + G_{zz}z^2\right] - \cdots - \sum_{r+s+t=n} G_{x^r y^s z^t}\frac{x^r y^s z^t}{r!s!t!} - \cdots, \qquad (2)$$

where $(H_0)_i$ $(i = x, y, z)$ are the Cartesian components of the magnetic field $\mathbf{H}_0 = -\nabla\Omega_0$ at the origin and the G_{ijk} $(i = x^r, j = y^s, k = z^t)$, with $r+s+t = n$, are elements of the n^{th} rank magnetic gradient tensor, each element comprising n^{th} derivatives of the scalar potential, evaluated at the origin. For the special case of a uniform gradient, the only non-zero terms are G_{ij} $(i, j = x, y, z)$, which are the components of the tensor of rank 2 that is usually meant by the term "gradient tensor" and is the quantity measured by most tensor gradiometers.

The tensor components are not all independent, because they are constrained by Maxwell's equations. In particular, the rank 2 gradient tensor is symmetric and traceless so there are only five independent tensor components, e.g. G_{xx}, G_{xy}, G_{xz}, G_{yy}, G_{yz}. Similarly, second order gradients of the field correspond to components of the third derivatives of the scalar potential, which define a rank 3 tensor G_{ijk} $(i,j,k = x,y,z)$ with 27 components. However, due to the constraints of symmetry and tracelessness of the rank 2 tensor, there are only seven independent components of G_{ijk}, e.g. G_{xxx}, G_{xyx}, G_{xzx}, G_{xyy}, G_{xyz}, G_{yyy}, G_{yyz} (Sui et al., 2017).

Taking the symmetry and tracelessness properties into account, for a uniform gradient equation (2) can be rewritten in terms of the five independent components listed above:

$$\Omega_0(x,y,z) = -(H_0)_x x - (H_0)_y y - (H_0)_z z - \frac{G_{xx}}{2}(x^2 - z^2) - G_{xy}xy - G_{xz}xz - \frac{G_{yy}}{2}(y^2 - z^2) - G_{yz}yz. \quad (3)$$

The form of equation (3) guarantees that $[G_{ij}]$ is symmetric and traceless. Each term associated with a vector component of the field on the RHS of (5) is a harmonic and homogeneous function of *x*, *y*, *z* of order 1; each term associated with a tensor component G_{ij} is also harmonic and is a homogeneous function of *x*, *y*, *z* of order 2.

PERMEABLE SPHERE IN AN ARBITRARY NON-UNIFORM FIELD

Consider a spherical homogeneous magnetisable body of radius *a*, with relative permeability μ_1, within an effectively infinite magnetisable medium with relative permeability μ_2. Assume that the applied field that would exist in the absence of the sphere arises from relatively distant sources that are all at a distance $r >> a$ from the origin, and that these sources are sufficiently distant that they are unaffected by the presence of the sphere. Then, in and around the sphere, the scalar potential Ω_0 associated with the arbitrary non-uniform applied field produced by these sources can be expressed in terms of a spherical harmonic expansion (Blakely, 1995, Chapter 6):

$$\Omega_0(r,\theta,\varphi) = \Omega_1 + \Omega_2 + \cdots = A_1 r S_1(\theta,\varphi) + A_2 r^2 S_2(\theta,\varphi) + \cdots + A_n r^n S_n(\theta,\varphi) + \cdots \quad (4)$$

where r,θ,ϕ are spherical polar co-ordinates with origin at the centre of the sphere, $S_n(\theta,\phi)$ denotes a spherical surface harmonic of degree *n*, $\Omega_n = r^n S_n(\theta,\phi)$ is the corresponding solid spherical harmonic of degree *n* that is regular at the origin, and the A_i are constants. Referring to equation (2), since $x = r\cos\phi \sin\theta$, $y = r\sin\phi \sin\theta$, $z = r\cos\theta$, it follows that the n^{th} order homogeneous function $f_n(x,y,z)$ is of the form $r^n g_n(\theta,\phi)$. Comparing (2) and (4), it is clear that the n^{th} order gradient tensor $\mathbf{G}_n$ is associated uniquely with the solid spherical harmonic $r^n S_n(\theta,\phi)$.

The resultant potential in response to the applied potential contribution Ω_n is:

$$\Omega'_n(r \geq a) = \Omega_n + A_n \frac{n(\mu_2-\mu_1)a^{2n+1}}{n\mu_1+(n+1)\mu_2}\frac{S_n(\theta,\varphi)}{r^{n+1}}, \quad (5a)$$

$$\Omega'_n(r \leq a) = \frac{(2n+1)\mu_2}{n\mu_1+(n+1)\mu_2}\Omega_n. \quad (5b)$$

The perturbed potential of (5a,b) obeys Laplace's equation, is regular at the origin, converges to the applied potential at large distances from the sphere and obeys the boundary conditions (continuity of the potential and of the normal component of **B** across the surface of the ellipsoid). From (5b) it follows that the n^{th} order gradient tensor inside the sphere is related to the unperturbed n^{th} order applied gradient tensor by:

$$\mathbf{G}'_n(r \leq a) = \frac{(2n+1)\mu_2}{n\mu_1+(n+1)\mu_2}\mathbf{G}_n. \quad (6)$$

Equation (6) extends the results of Clark (2018) for spheres to higher order gradients. For a spherical cavity within an infinite permeable medium (μ_1 = 1; $\mu_2 = 1+\chi$), the internal n^{th} order gradient tensor is amplified with respect to the applied n^{th} order gradient tensor:

$$\mathbf{G}'_n(r \leq a;\ \mu_1 = 1, \mu_2 = 1+\chi) = \frac{(1+\chi)}{1+(n+1)\chi/(2n+1)}\mathbf{G}_n. \quad (7)$$

For a spherical permeable body ($\mu_1 = 1+\chi$) within an infinite nonmagnetic medium (μ_2 = 1), the internal n^{th} order gradient tensor is somewhat shielded with respect to the applied gradient:

$$\mathbf{G}'_n(r \leq a;\ \mu_1 = 1+\chi, \mu_2 = 1) = \frac{1}{1+n\chi/(2n+1)}\mathbf{G}_n. \quad (8)$$

Thus the shielding factors for n^{th} order gradients inside a magnetisable sphere, $1/[1+n\chi/(2n+1)]$, differ from the shielding factor for the applied field, which is $1/(1+\chi/3)$. The shielding factor for the rank 2 gradient tensor is $1/(1+2\chi/5)$.

PERMEABLE ELLIPSOID IN A GRADIENT

Consider a homogeneous magnetisable ellipsoid, with semi-axes $a > b > c$ along Cartesian axes x_1, x_2, x_3 with relative permeability μ_1, within an effectively infinite magnetisable medium with relative permeability μ_2. Each point (x_1, x_2, x_3) is associated with ellipsoidal co-ordinates (ρ, μ, ν) (Dassios, 2012). A non-uniform applied field has an associated potential given by:

$$\Omega_0(x_1,x_2,x_3) = -(H_0)_1 x_1 - (H_0)_2 x_2 - (H_0)_3 x_3 - \frac{G_{11}}{2}({x_1}^2 - {x_3}^2) - G_{12}x_1x_2 - G_{13}x_1x_3 - \frac{G_{yy}}{2}({x_2}^2 - {x_3}^2) - G_{23}x_2x_3 + \cdots. \quad (9)$$

which can be expressed as an ellipsoidal harmonic expansion:

$$\Omega_0(\rho,\mu,\nu) = \sum_{n=1}^{\infty}\sum_{m=1}^{2n+1} C_n^m \mathbf{E}_n^m(\rho,\mu,\nu). \quad (10)$$

$$\mathbf{E}_n^m(\rho,\mu,\nu) = E_n^m(\rho)E_n^m(\mu)E_n^m(\nu) = E_n^m(\rho)S_n^m(\mu,\nu) \quad (11)$$

In (10) each $\mathbf{E}_n^m$ is an interior ellipsoidal harmonic (ellipsoidal harmonic of the first kind) of degree *n* and order *m*, regular at the origin. The E_n^m are Lamé functions (solutions of Lamé's differential equation) and $S_n^m(\mu,\nu) = E_n^m(\mu)E_n^m(\nu)$ is an ellipsoidal surface harmonic of degree *n* and order *m*.

The coefficients of the three harmonics of degree 1 in the expansion are proportional to the three field components at the origin, the coefficients of the five harmonics of degree 2 are related to the five independent components of the rank 2 gradient tensor, and so on. When the ellipsoid is inserted into a field with a uniform gradient, the internal and external perturbed potentials take the form:

$$\Omega'(\rho \leq a) = A_1^1\mathbf{E}_1^1 + A_1^2\mathbf{E}_1^2 + A_1^3\mathbf{E}_1^3 + A_2^1\mathbf{E}_2^1 + A_2^2\mathbf{E}_2^2 + A_2^3\mathbf{E}_2^3 + A_2^4\mathbf{E}_2^4 + A_2^5\mathbf{E}_2^5, \quad (12)$$

$$\Omega'_n(\rho \geq a) = \Omega_0 + B_1^1\mathbf{F}_1^1 + B_1^2\mathbf{F}_1^2 + B_1^3\mathbf{F}_1^3 + B_2^1\mathbf{F}_2^1 + B_2^2\mathbf{F}_2^2 + B_2^3\mathbf{F}_2^3 + B_2^4\mathbf{F}_2^4 + B_2^5\mathbf{F}_2^5, \quad (13)$$

where the $\mathbf{F}_n^m$ are exterior ellipsoidal harmonics (ellipsoidal harmonics of the second kind) of degree *n* and order *m*, regular at infinity. They are given by:

$$\mathbf{F}_n^m(\rho,\mu,\nu) = F_n^m(\rho)E_n^m(\mu)E_n^m(\nu) = F_n^m(\rho)S_n^m(\mu,\nu) = (2n+1)E_n^m(\rho)I_n^m(\rho)S_n^m(\mu,\nu), \quad (14)$$

where

$$I_n^m(\rho) = \int_\rho^\infty \frac{dt}{[E_n^m(t)]^2\sqrt{t^2-a^2+b^2}\sqrt{t^2-a^2+c^2}}. \quad (15)$$

The boundary conditions must be satisfied across the ellipsoid surface, independently for each surface harmonic term $S_n^m(\mu,\nu)$. The solution of the boundary value problem for the interior and exterior potentials yields for the unknown constants:

$$A_n^m = \frac{C_n^m}{1+\frac{(\mu_1-\mu_2)}{\mu_2}D_n^m}, \quad (16)$$

$$B_n^m = \frac{C_n^m(\mu_1-\mu_2)\,(E_n^m)'(a)}{1+\frac{(\mu_1-\mu_2)}{\mu_2}D_n^m}, \quad (17)$$

where

$$D_n^m = -\frac{(E_n^m)'(a)\,I_n^m(a)}{E_n^m(a)\,(I_n^m)'(a)} = bcE_n^m(a)(E_n^m)'(a)\,I_n^m(a). \quad (18)$$

The prime $'$ here indicates the derivative with respect to ρ.

Thus the internal potential gradient characterised by an *ellipsoidal harmonic* of degree and order *n*, *m* has the same form as the applied potential gradient associated with the *harmonic* of that degree and order. However each ellipsoidal harmonic contribution to the internal gradient is attenuated or enhanced by a factor that depends on (i) the permeability contrast (or equivalently the susceptibility contrast) between the ellipsoid and the surrounding medium, divided by the permeability of the surrounding medium, and (ii) a geometric factor D_n^m that depends on the axial ratios of the ellipsoid and the degree and order of the applied gradient.

D_n^m can be regarded as a type of demagnetising factor for the potential gradient term of degree and order *n*, *m*. These factors are not all independent. Higher order demagnetising factors can be expressed in terms of the conventional demagnetising factors D_1, D_2, D_3 (see Table 1). The demagnetising factors of orders 1, 2 and 3 obey the following relationships:

$$\sum_{m=1}^{3} D_1^m = 1;\ \sum_{m=1}^{5} D_2^m = 2;\ \sum_{m=1}^{7} D_3^m = 3. \quad (19)$$

Figure 1 illustrates the dependence of the second order factors on the ellipsoid shape.

The application of (16) to the off-diagonal gradient tensor components, which are each associated with unique ellipsoidal harmonics, is straightforward. These components are shielded (if $\mu_1 > \mu_2$) or amplified (if $\mu_1 < \mu_2$) by a simple factor. The internal gradient components are given by:

$$G'_{12} = \frac{G_{12}}{1+\frac{(\mu_1-\mu_2)}{\mu_2}D_2^3},\ G'_{13} = \frac{G_{13}}{1+\frac{(\mu_1-\mu_2)}{\mu_2}D_2^4},\ G'_{23} = \frac{G_{23}}{1+\frac{(\mu_1-\mu_2)}{\mu_2}D_2^5}. \quad (20)$$

The situation is more complicated for the diagonal components of the gradient tensor, because the ellipsoidal harmonic terms correspond to combinations of these tensor components, rather than to individual diagonal components. For example, the diagonal tensor components G_{11} and G_{22} both contribute $\mathbf{E}_1^1$ and $\mathbf{E}_1^2$ terms to the ellipsoidal harmonic expansion of the potential. The internal, demagnetisation- affected, diagonal tensor components are related to the applied components by:

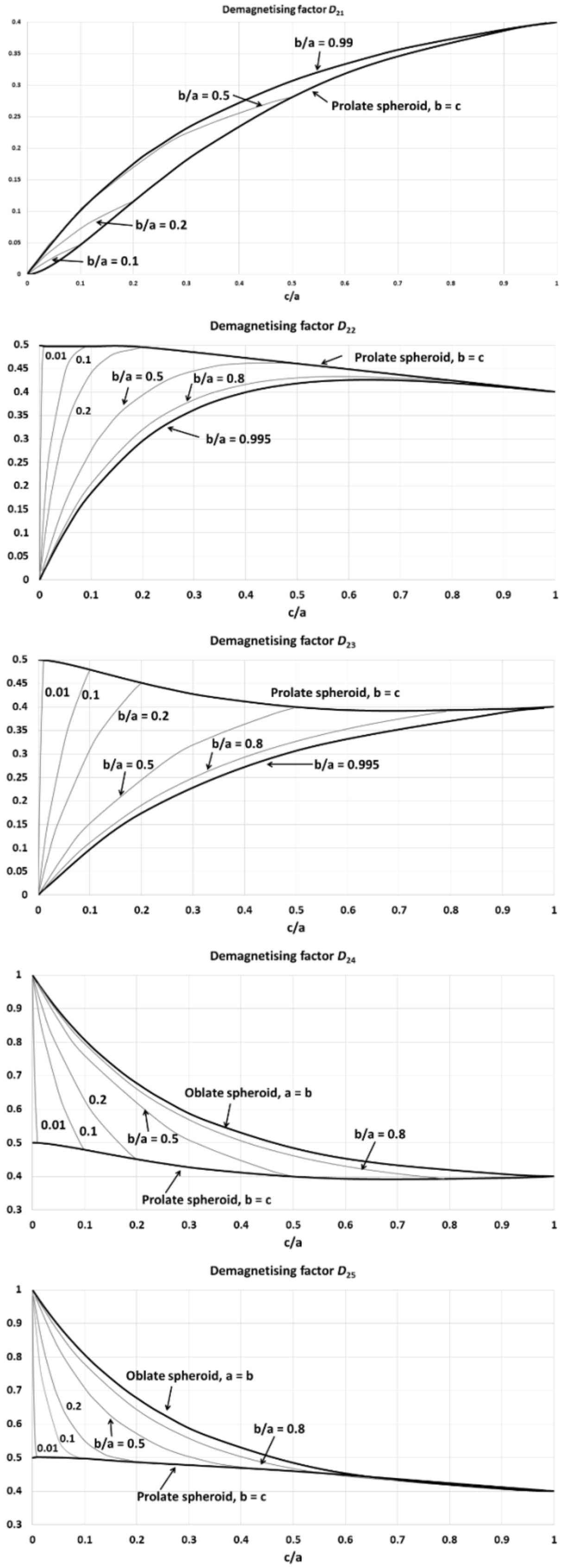

Figure 1. Second order demagnetising factors of triaxial ellipsoids as a function of ellipsoid shape. Note that the convergence of the curves on the right hand side of the plots

at c/a =1, corresponds to a sphere with an isotropic second order demagnetising factor of 2/5 = 0.4.

$$\begin{bmatrix} G'_{11} \\ G'_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} G_{11} \\ G_{22} \end{bmatrix} = \mathbf{M} \begin{bmatrix} G_{11} \\ G_{22} \end{bmatrix}, \quad (21)$$

where the elements of the matrix **M** are:

$$c_{11} = \frac{-(a^2-c^2)}{3(\Lambda-\Lambda')} \left[\frac{\Lambda-b^2}{(\Lambda-a^2)\left[1+\frac{(\mu_1-\mu_2)}{\mu_2} D_2^1\right]} - \frac{\Lambda'-b^2}{(\Lambda'-a^2)\left[1+\frac{(\mu_1-\mu_2)}{\mu_2} D_2^2\right]} \right], \quad (22)$$

$$c_{12} = \frac{-(b^2-c^2)}{3(\Lambda-\Lambda')} \left[\frac{1}{\left[1+\frac{(\mu_1-\mu_2)}{\mu_2} D_2^1\right]} - \frac{1}{\left[1+\frac{(\mu_1-\mu_2)}{\mu_2} D_2^2\right]} \right], \quad (23)$$

$$c_{21} = \frac{-(a^2-c^2)}{3(\Lambda-\Lambda')} \left[\frac{1}{\left[1+\frac{(\mu_1-\mu_2)}{\mu_2} D_2^1\right]} - \frac{1}{\left[1+\frac{(\mu_1-\mu_2)}{\mu_2} D_2^2\right]} \right], \quad (24)$$

$$c_{22} = \frac{-(b^2-c^2)}{3(\Lambda-\Lambda')} \left[\frac{\Lambda-a^2}{(\Lambda-b^2)\left[1+\frac{(\mu_1-\mu_2)}{\mu_2} D_2^1\right]} - \frac{\Lambda'-a^2}{(\Lambda'-b^2)\left[1+\frac{(\mu_1-\mu_2)}{\mu_2} D_2^2\right]} \right]. \quad (25)$$

The constants Λ, Λ' used above and in Table 1 are the solutions of:

$$\frac{1}{\Lambda-a^2} + \frac{1}{\Lambda-b^2} + \frac{1}{\Lambda-c^2} = 0, \quad (26)$$

which are given by (Dassios, 2012):

$$\Lambda = \frac{1}{3}(a^2+b^2+c^2) + \frac{1}{3}\sqrt{a^4+b^4+c^4-a^2b^2-b^2c^2-a^2c^2}, \quad (27)$$

$$\Lambda' = \frac{1}{3}(a^2+b^2+c^2) - \frac{1}{3}\sqrt{a^4+b^4+c^4-a^2b^2-b^2c^2-a^2c^2}, \quad (28)$$

For specified shape and orientation of the ellipsoid and specified susceptibilities of the ellipsoid and its host rock, the imposed gradient tensor elements can be obtained from the internal gradient tensor elements by calculating the demagnetising factors, using the formulae given in Table 1, and inverting (20)-(21):

$$\begin{bmatrix} G_{11} \\ G_{22} \end{bmatrix} = \mathbf{M}^{-1} \begin{bmatrix} G'_{11} \\ G'_{22} \end{bmatrix}, \quad (29)$$

$$G_{12} = G'_{12}\left[1+\frac{(\mu_1-\mu_2)}{\mu_2} D_2^3\right], G_{13} = G'_{13}\left[1+\frac{(\mu_1-\mu_2)}{\mu_2} D_2^4\right], G_{23} = G'_{23}\left[1+\frac{(\mu_1-\mu_2)}{\mu_2} D_2^5\right]. \quad (30)$$

CORRECTION OF BOREHOLE MEASUREMENTS

I now consider the case of an ellipsoid penetrated by a borehole. Define a new Cartesian co-ordinate system with the *z* axis down the hole, *x* perpendicular to the hole and upward-directed in the vertical plane, with *y* oriented to form a right-handed system. Clark (2018) showed that, in terms of the tensor $\mathbf{G}'$ measured within the cylindrical cavity and with respect to the borehole co-ordinate system, the gradient tensor **G** in the magnetic material around the hole is given by:

$$\begin{bmatrix} \frac{(1+\chi/2)G'_{xx}-\chi G'_{zz}/4}{1+\chi} & \frac{(1+\chi/2)G'_{xy}}{1+\chi} & \frac{(1+\chi/2)G'_{xz}}{1+\chi} \\ \frac{(1+\chi/2)G'_{xy}}{1+\chi} & \frac{(1+\chi/2)G'_{yy}-\chi G'_{zz}/4}{1+\chi} & \frac{(1+\chi/2)G'_{yz}}{1+\chi} \\ \frac{(1+\chi/2)G'_{xz}}{1+\chi} & \frac{(1+\chi/2)G'_{yz}}{1+\chi} & G'_{zz} \end{bmatrix} \quad (31)$$

The expression (31) enables the gradient tensor in the medium around the borehole, e.g. in the ellipsoidal body penetrated by the borehole, to be estimated from measurements made in the borehole. This gradient tensor is related to the external gradients as described in the previous sections.

SIGNIFICANCE OF INTERNAL GRADIENTS

Measurements of magnetic gradients within a borehole that penetrates a homogeneous magnetic source depend on: (i) the imposed gradient due to the geomagnetic core field, regional trends and nearby external sources, (ii) the gradients that arise from the remanent magnetisation contrast and permeability contrast across the boundary of the magnetic source, i.e. from the boundaries of the penetrated source and (iii) from the effect of the borehole cavity, which reflects the magnetic property contrasts across the borehole boundary.

Homogeneous ellipsoids have the special property that the character of the applied field is retained by the internal field: a uniform applied field results in a uniform internal field, a uniform applied gradient results in a uniform internal gradient etc. On the other hand the boundaries of homogeneous bodies with angular interfaces, such as sheets, prisms and polyhedra, produce internal gradients that supplement imposed gradients from external sources. These non-uniform demagnetising fields are strongest near corners and vertices. Bodies with irregular boundaries produce strong gradients near the boundaries. However the borehole cavity field correction can still be applied to borehole data that lies well within a magnetic body. In that case the measured gradient provides information about the geometry of the bounding surface.

A simple model that illustrates the effects of imposed gradients on measured gradient anomalies inside and outside a magnetic source is shown in Figure 2. A deep, strongly magnetic source imposes a substantial gradient on a shallow magnetic source in the form of a spherical orebody with susceptibility 0.5 SI, which corresponds to about 12 volume % magnetite. The internal gradients are modified by self-demagnetisation; the external anomaly is affected by quadrupole + octupole etc. contributions that arise from rank 2 + rank 3 applied gradient tensors, which add to the dipole contribution from the average magnetisation of the sphere.

CONCLUSIONS

Demagnetising factors differ for applied field components and components of the applied gradient tensor. For a permeable sphere the effects of self-demagnetisation on the applied rank 2 gradient tensor are isotropic and all tensor components have the same second order demagnetising factor of 0.4, compared to the first order demagnetising factor for field components, which is 1/3. The effects of self-demagnetisation on the imposed gradient tensor are anisotropic for an ellipsoidal body. For a permeable ellipsoid in a non-magnetic host rock, effects of self-demagnetisation cause shielding of off-diagonal tensor components (defined with respect to the ellipsoid axes), with a simple shielding factor that depends on the axial ratios of the ellipsoid and its susceptibility. By contrast self-demagnetisation causes mixing of the applied diagonal tensor components. Field and gradient components measured in a borehole should be corrected for the cavity effects when the borehole penetrates a magnetic body. The corrected components can then be interpreted in terms of the magnetic environment, which includes external sources and the boundary of the penetrated body. For ellipsoids, including limiting cases such as spheres, spheroids, circular and elliptical cylinders, sheets etc., boundary effects can be incorporated via generalised demagnetising factors.

Table 1. First and second order demagnetising factors for triaxial ellipsoids

Demagnetising factor D_n^m
$D_1^1 = D_1 = \frac{abc}{(a^2-b^2)\sqrt{a^2-c^2}} \times [F(\varphi,\alpha) - E(\varphi,\alpha)]$
$D_1^2 = D_2 = \frac{abc\sqrt{a^2-c^2}}{(a^2-b^2)(b^2-c^2)} \times \left[E(\varphi,\alpha) - \frac{(b^2-c^2)}{(a^2-c^2)}F(\varphi,\alpha) - \frac{c(a^2-b^2)}{ab\sqrt{a^2-c^2}}\right]$
$D_1^3 = D_3 = abcI_1^3(a) = \frac{abc}{(b^2-c^2)\sqrt{a^2-c^2}}\left[\frac{b\sqrt{a^2-c^2}}{ac} - E(\varphi,\alpha)\right]$
$D_2^1 = 1 - \Lambda\left(\frac{D_1}{\Lambda-a^2} + \frac{D_2}{\Lambda-b^2} + \frac{D_3}{\Lambda-c^2}\right)$
$D_2^2 = 1 - \Lambda'\left(\frac{D_1}{\Lambda'-a^2} + \frac{D_2}{\Lambda'-b^2} + \frac{D_3}{\Lambda'-c^2}\right)$
$D_2^3 = \left(\frac{a^2+b^2}{a^2-b^2}\right)(D_2 - D_1)$
$D_2^4 = \left(\frac{a^2+c^2}{a^2-c^2}\right)(D_3 - D_1)$
$D_2^5 = \left(\frac{b^2+c^2}{b^2-c^2}\right)(D_3 - D_2)$

$$\varphi = sin^{-1}\sqrt{\frac{a^2-c^2}{\rho}};\ \alpha = sin^{-1}\sqrt{\frac{a^2-b^2}{a^2-c^2}}. \quad (32)$$

In Table 1 $F(\varphi, k)$ and $E(\varphi, k)$ are, respectively, incomplete elliptic integrals of the first and second kinds, with amplitude φ and modular angle α given by:

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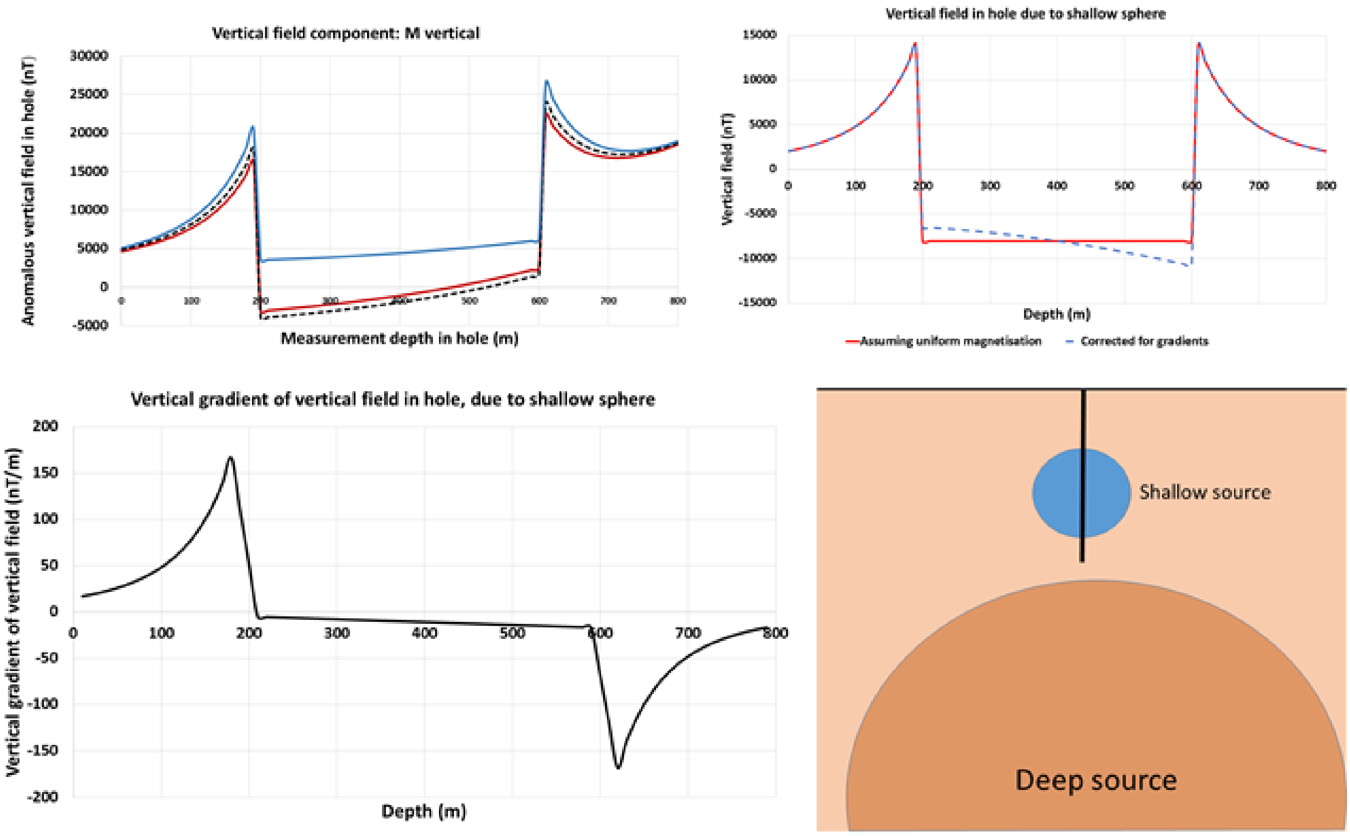

Figure 2. Effects of imposed gradient on magnetic field and gradient anomalies in a vertical borehole that penetrates a strongly magnetic spherical orebody. The geomagnetic field is vertical and a strong magnetic source that underlies and imposes a substantial gradient on the orebody. Top Left: The blue curve shows the corrected borehole profile of the vertical field component, including the effects of non-uniform magnetisation on the internal field and a quadrupole contribution to the external field. The red curve shows the predicted vertical field assuming uniform induced magnetisation in the regional geomagnetic field; the dashed black curve modifies this prediction by using the local field intensity, affected by the deep source, rather than the IGRF field. Top Right: The vertical field due to the shallow sphere, after removal of the background gradient due to the deep source. Neglecting the applied gradient, one would infer a uniform magnetisation, which would produce a uniform internal field. The applied gradient produces a non-uniform magnetisation and an internal gradient that is evident in the blue dashed curve. The quadrupole and octupole contributions to the external anomalous field of the sphere are measurable, but not visible at this scale. Bottom Left: Vertical gradient of the vertical field due to the shallow sphere. This would correspond to the gradient that would be measured by a tensor gradiometer in the borehole, after removal of the trend due to the deep source. Bottom Right: schematic view of the sources used for this simple model.